**S₂ Ideals In Commutative Rings**

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**ABSTRACT**

Let R be a commutative ring with identity, let a be a nonzero element in R. In (Al-Taha, 2011) we give the definition of s₁ ideal and we study some of its properties, while this paper deals with a new definition for the principal ideal I=<a> of the ring R that we call s₂ ideal, we give some results about s₂ ideals also we give the relation between s₂ ideals and s₁ ideals. We prove the following result among others the ideal I=<a> is s₂ ideal of R if, and only if I=<a> is s₂ ideal of R[x₁, x₂, ..., xₙ].

1. **Introduction**

Let R be a commutative ring with identity, let a be a nonzero element in R, the principal ideal I=<a> is called s₂ ideal if, and only if the following holds:

If \( ab = a, \ b \in R \) and \( b \neq 1 \)

There exists \( a' \in R, \ a' \neq 0 \) such that:

\( a'b = aa' \)  

This definition will serve as our main tool throughout this paper. Recall that: The principal ideal I=<a> is called s₁ ideal if, and only if the following holds:

If \( ab = a, \ b \in R \)

There exists \( a'^\prime \in R, \ a'^\prime \neq 0 \) such that:

\( a'b = 0 \)  

(Al-Taha, 2011)

Throughout this work we use the following notations:

R: Commutative ring with identity.

R[x]: The ring of polynomials over R in the indeterminate x.

R[x₁, x₂, ..., xₙ]: The ring of polynomials over R in the indeterminates x₁, x₂, ..., xₙ.

\( \text{ann}(a) \): Annihilator of an element \( a \in R = \{ x \in R: ax = 0 \} \).
2. Main results

In this section we state down the following results:

Theorem (1): Let R be a commutative ring with identity, let (a) be a nonzero element of R. If I = <a> is s₁ ideal of R then it is s₂ ideal of R.

Theorem (2): Let R be a commutative ring with identity, let (a) be a nonzero element of R. If a is an idempotent element of R, then I = <a> is s₂ ideal of R.

Theorem (3): Let R₁, R₂ be any two commutative rings with identity, let <a₁> and <a₂> be any two s₂ ideals of R₁ and R₂ respectively then <(a₁,a₂)> is s₂ ideal of the direct sum of the rings R₁ ⊕ R₂.

Corollary (1): Let R₁, R₂, ..., Rₙ be n commutative rings with identity, let <a₁>, <a₂>, ..., <aₙ> be n s₂ ideals of R₁, R₂, ..., Rₙ respectively then <(a₁,a₂,...,aₙ)> is S₂ ideal of the direct sum of the rings R₁ ⊕ R₂ ⊕ ... ⊕ Rₙ.

Theorem (4): Let R be a commutative ring with identity, <a> is s₂ ideal of R if, and only if <a> is s₂ ideal of R[x].

Corollary (2): Let R be a commutative ring with identity, <a> is s₂ ideal of R if and only if <a> is s₂ ideal of R[x₁, x₂, ..., xₙ].

Theorem (5): Let R be a commutative ring with identity, let a be a nonzero element in R. If <a+I> is s₂ ideal of the quotient ring R/I, where I = ann(a) then <a> is s₂ ideal of R.

Theorem (6): Let R be a commutative ring with identity, let a be a nonzero element in R. If a is an idempotent element of R then <a+I> is s₂ ideal of the quotient ring R/I where I is an ideal of R.

Theorem (7): Let R and R' be any two commutative rings with identity, let φ be an isomorphism between these two rings. <a> is s₂ ideal of R if, and only if <φ(a)> is s₂ ideal R'.

3. Proofs

In this section we prove the main results.

Proof of theorem (1):

Let <a> be s₁ ideal of R to show that <a> is s₂ ideal of R, let
ab = a, be R
Since <a> is s₁ ideal of R, using (4) we get there exists a'ε R, a'≠0 such that:
a'b=0  \hspace{1cm} (5)
aa'b =a'a=o, hence
a'b =a'a  \hspace{1cm} (6)
Thus <a> is s₂ ideal

The next example shows that s₂ ideal need not be s₁ ideal.

Example: The ideal <4> in the ring Z₁₂ is s₂ ideal, which is not s₁ ideal.
Proof of theorem (2):

Assume that \( a \) is nonzero idempotent element in the ring \( R \). (Gallian, 2006)
To show that \( \langle a \rangle \) is \( s_2 \) ideal of the ring \( R \), let
\[ ab = a, \quad b \in R, \ b \neq 1. \]
But \( a \) is an idempotent element in \( R \), we get
\[ ab = aa \] (7)
So, there exists \( a' = a \) using (2) we get \( \langle a \rangle \) is \( s_2 \) ideal of \( R \).

Proof of theorem (3):

Suppose that \( \langle a_1 \rangle \) and \( \langle a_2 \rangle \) are \( s_2 \) ideals of the rings \( R_1 \) and \( R_2 \) respectively our task is to show that the principal ideal \( \langle (a_1, a_2) \rangle \) is \( s_2 \) ideal of the direct sum of rings \( R_1 \oplus R_2 \), let \( (a_1, a_2)(b_1, b_2) = (a_1, a_2) \) where \( (b_1, b_2) \in R_1 \oplus R_2 \) and \( (b_1, b_2) \neq (1, 1) \).
We get \( a_1 b_1 = a_1 \) and \( a_2 b_2 = a_2 \). (Gallian, 2006)
Since \( \langle a_1 \rangle \) and \( \langle a_2 \rangle \) are \( s_2 \) ideals of the rings \( R_1 \) and \( R_2 \) respectively using (2), we get there exist \( a'_1 \) and \( a'_2 \) nonzero elements in \( R_1 \) and \( R_2 \) such that:
\[ a_1' b_1 = a_1' a_1 \quad \text{and} \quad a_2' b_2 = a_2' a_2 \] (8)
\((a_1', a_2')(b_1, b_2) = (a_1', a_2')(a_1, a_2)\), this end the first half of the theorem.

To prove the other half of the theorem, suppose that \( \langle (a_1, a_2) \rangle \) is \( s_2 \) ideal of \( R_1 \oplus R_2 \), to show that \( \langle a_1 \rangle, \langle a_2 \rangle \) are \( s_2 \) ideals of \( R_1 \) and \( R_2 \) respectively, let
\[ a_1 b_1 = a_1 \quad \text{and} \quad a_2 b_2 = a_2 \quad \text{where} \quad b_1 \in R_1, \ b_1 \neq 1 \quad \text{and} \quad b_2 \in R_2, \ b_2 \neq 1. \]
We get,
\[ (a_1, a_2)(b_1, b_2) = (a_1, a_2) \] but \( \langle (a_1, a_2) \rangle \) is \( s_2 \) ideal of \( R_1 \oplus R_2 \), using (2) there exists \( (a_1', a_2') \) nonzero element in \( R_1 \oplus R_2 \) such that:
\[ (a_1', a_2')(b_1, b_2) = (a_1', a_2')(a_1, a_2) \] (9)
This implies that:
\[ a_1' b_1 = a_1' a_1 \quad \text{and} \quad a_2' b_2 = a_2' a_2. \]
Thus \( \langle a_1 \rangle, \langle a_2 \rangle \) are \( s_2 \) ideals of \( R_1 \) and \( R_2 \).

Proof of theorem (4):

Suppose that \( \langle a \rangle \) is \( s_2 \) ideal of \( R \), to show that \( \langle a \rangle \) is \( s_2 \) ideal of \( R[x] \).
Let \( af(x) = a, \ f(x) = b_0 + b_1 x + \ldots + b_n x^n \in R[x] \) and \( f(x) \neq 1 \)
\[ ab_n = a \quad \text{(Herstein, 1990)} \]
Since \( \langle a \rangle \) is \( s_2 \) ideal of \( R \), using (2) we get there exists \( a' \in R \) such that
\[ a'b_0 = a'a \] , it is clear that:

\[ \mathfrak{a}\mathfrak{a}' f(x) = \mathfrak{a}\mathfrak{a}' a \quad (10) \]

Thus \(<a>\) is \(s_2\) ideal of \(R[x]\).

Conversely, assume that \(<a>\) is \(s_2\) ideal of \(R[x]\) to show that \(<a>\) is \(s_2\) ideal of \(R\), let \(ab = a, \quad b \in R, b \neq 1\).

Since \(<a>\) is \(s_2\) ideal of \(R[x]\), using (2) we get there exists \(f(x) \in R[x]\)

\[ f(x) = a_0 + a_1 x + \ldots + a_n x^n \] and

\[ f(x)b = f(x)a \] this implies that:

\[ a'b = a'a \quad (11) \]

Therefore \(<a>\) is \(s_2\) ideal of \(R\).

**Proof of theorem (5):**

Suppose that \(<a+I>\) is \(s_2\) ideal of \(R/I\). To show that \(<a>\) is \(s_2\) ideal of \(R\), let

\[ ab = a, \quad b \in R, b \neq 1. \]

\[ (a+I)(b+I) = (a+I) \]

since \(<a+I>\) is \(s_2\) ideal of \(R/I\), using (2) we get there exists \((a'+I) \in R/I\) such that:

\[ (a'+I)(b+I) = (a'+I)(a+I) \]

\[ (ab+I) = (aa+I) \]

\[ a'b - a'a \in I \] this means that:

\[ \mathfrak{a}\mathfrak{a}' b = \mathfrak{a}\mathfrak{a}' a \] . Thus \(<a>\) is \(s_2\) ideal of \(R\).

**Proof of theorem (6):**

Suppose that \(a\) is nonzero idempotent element in \(R\), to show that \(<a+I>\) is \(s_2\) ideal of \(R/I\), let

\[ (a+I)(b+I) = (a+I) \quad (12) \]

\[ (a+I)(b+I) = (a+I)(a+I) \] since \(a\) is idempotent element of \(R\) (Kaplansky, 1974; Blyth & Robertson 1984). We get that:

\(<a+I>\) is \(s_2\) ideal of \(R/I\).

**Proof of theorem (7):**

Suppose that \(\varphi\) is an isomorphism from \(R\) into \(R'\) and that \(<a>\) is \(s_2\) ideal of \(R\), to show that

\(<\varphi(a)>\) is \(s_2\) ideal of \(R'\) (Blyth & Robertson, 1984)

Let \(\varphi(a)\varphi(b) = \varphi(a)\), we get
ab =a
Since $<a>$ is $s_2$ ideal of R using (2) we get there exists $a' \epsilon R$ , $a' \neq 0$ such that:

$$a'b = a'a$$  \hspace{1cm} (13)

This implies that:

$$\phi(a')\phi(b) = \phi(a')\phi(a)$$

Thus $<\phi(a)>$ is $s_2$ ideal of $R'$. 

Conversely: Assume that $<\phi(a)>$ is $s_2$ ideal of $R'$, to show that $<a>$ is $s_2$ ideal of R let

$$ab = a$$ this implies that

$$\phi(a)\phi(b) = \phi(a)$$

Since $\phi(a)$ is $s_2$ ideal of $R'$ using (2) we get there exists $\phi(a') \epsilon R'$ $\phi(a') \neq 0$ such that;

$$\phi(a') \phi(b) = \phi(a') \phi(a)$$  \hspace{1cm} (14)

this implies that there exists $a' \epsilon R$ such that:

$$a'b = a'a$$

Thus $<a>$ is $s_2$ ideal of R.

References


